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Adiabatic Decay of Internal Solitons in a Rotating Ocean

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Abstract

The adiabatic decay of different types of internal wave solitons caused by the Earth' rotation is considered within the framework of the Gardner–Ostrovsky equation. The governing equation describing such processes includes quadratic and cubic nonlinear terms, as well as the Boussinesq and Coriolis dispersions. It is shown that at the early stage of evolution solitons gradually decay under the influence of weak Earth' rotation. The characteristic decay time is derived for different types of solitons.

Introduction

The model Gardner–Ostrovsky (GO) equation, was derived for the description of long internal waves [7, 9]:

$$\frac{\partial}{\partial x}\left[\frac{\partial u}{\partial t}+\left(c+\alpha u+\alpha_1 u^2\right)\frac{\partial u}{\partial x}+\beta\frac{\partial^3 u}{\partial x^3}\right]=\gamma u, \tag{1}$$

where c is the velocity of dispersionless linear waves, α and α_1 are the coefficient of quadratic and cubic nonlinearities, respectively, and β and γ are the coefficients of small-scale (Boussinesq) and large-scale (Coriolis) dispersion, respectively. The variable $u(x, t)$ describes a perturbation of an isopycnal surface (the surface of equal density) from its rest position. The coefficients of equation (1) are well known and can be found both for waves in continuously stratified fluid and for interfacial waves in two-layer fluid (see, e.g., [1, 6, 8]).

Equation (1) is apparently non-integrable and even its stationary solutions are unknown. In the meantime, in the absence of rotation the GO equation reduces to the well-known and completely integrable Gardner equation [11, 12]. The latter equation has soliton solutions whose shape depends on the amplitude and sign of the cubic coefficient α_1. It is a matter of interest to study the influence of weak rotation on the dynamics of quasi-stationary Gardner solitons in application to large amplitude internal waves. Such waves are often observed in shallow coastal regions where they may have an influence on the human activity, engineering constructions off-shore petroleum exploration, production and sub-sea storage activities, etc.

In this paper we present asymptotic solutions for the slowly varying Gardner solitons of internal waves due to influence of Earth' rotation. We show that the rotation leads to soliton terminal decay; we estimate the life time of Gardner solitons and show that the character of soliton decay is different for the bell-shaped and table-top solitons having plane maximums.

To obtain an asymptotic solution to the GO equation (1) it is convenient to present it in the dimensionless form using the normalized variables:

$$\xi=\frac{x-ct}{L_0};\quad \tau=t\frac{\alpha U_0}{L_0};\quad \upsilon=\frac{u}{U_0}, \tag{2}$$

where U_0 is the characteristic amplitude, and L_0 is the characteristic width of the initial perturbation. In new variables equation (1) is:

$$\frac{\partial\upsilon}{\partial\tau}+\left(\upsilon+\mu\upsilon^2\right)\frac{\partial\upsilon}{\partial\xi}+\frac{1}{\mathrm{Ur}}\frac{\partial^3\upsilon}{\partial\xi^3}=-\varepsilon\int_{\xi}^{+\infty}\upsilon\left(\xi'\right)d\xi', \tag{3}$$

where $\mu = \alpha_1 U_0/\alpha$, $\varepsilon = \gamma L_0^2/(\alpha U)$, and $\mathrm{Ur} = \alpha U_0 L_0^2/\beta$ is the well-known Ursell parameter in the theory of shallow water waves (see, e.g., [2]).

Equation (1) should be augmented by the initial condition. Here we are interested in the evolution of solitary waves, therefore we assume that the initial condition can be presented in terms of a pulse-type function: $u(0, x) = U_0 F(x/L_0)$, where a dimensionless function $F(\xi)$ has a unit height and width. In the dimensionless variables, the initial condition for equation (3) takes a simple form $\upsilon(0, \xi) = F(\xi)$.

In the typical oceanic conditions the coefficients of equation (1) are such that $\alpha < 0$, $\beta > 0$, $\gamma > 0$, whereas the coefficient α_1 may be both positive and negative (see, e.g., [1]). Therefore the dimensionless coefficient μ may be also both positive and negative, whereas the parameters Ur and ε in equation (3) are always positive for oceanic waves.

Dynamics of Gardner solitons when $\mu < 0$ ($\alpha_1 < 0$)

We start our analysis with the most typical oceanic case when α_1 is negative (and hence μ is negative as well). For the near-surface pycnocline an initial perturbation is a soliton of negative polarity, whereas for the near-bottom pycnocline the initial soliton has a positive polarity [1].

When $\varepsilon = 0$, equation (3) reduces to the well-known Gardner equation. One of the exact stationary solutions to this equation is the so-called "fat" soliton [1, 9]:

$$\upsilon=A\left[1+B\cosh\left(\frac{\xi-V\tau}{\Delta/2L_0}\right)\right]^{-1}, \tag{4}$$

where $0 \le B \le 1$, and all other parameters can be presented in terms of B:

$$A=\frac{1-B^2}{-\mu},\quad \Delta=L_0\sqrt{\frac{-24\mu}{\mathrm{Ur}\left(1-B^2\right)}},\quad V=\frac{A}{6}. \tag{5}$$

The amplitude of the Gardner soliton is determined by the formula

$$U=\frac{A}{1+B}U_0\equiv-\frac{1-B}{\mu}U_0. \tag{6}$$

The shape of the Gardner soliton varies with B from the bell-shaped KdV soliton, when $B \to 1$, to the table-top soliton, when $B \to 0$. In the KdV limit ($B \to 1$) solution (4) reduces to

$$\upsilon = \frac{A}{2}\cosh^{-2}\left(\frac{\xi - V\tau}{\Delta/L_0}\right). \qquad (7)$$

If $\varepsilon \neq 0$, but sufficiently small, $\varepsilon << 1$, then solution (4) is no longer valid. However, if the Gardner soliton is structurally stable, then under the influence of a small perturbative term in the right-hand side of equation (3) it may experience just a gradual adiabatic variation with time, keeping the shape and the relationships between other parameters as per equations (5) and (6) at any time. Then the evolution of soliton parameters with time can be calculated with the help of the perturbation theory described in many papers (see, e.g., [3, 7]). Here we apply a similar approach to calculate time variation of GO soliton shape and parameters under the influence of fluid rotation.

Multiplying equation (3) by υ and integrating it with respect to ξ in the infinite limits, we obtain the energy balance equation:

$$\frac{d}{d\tau}\int_{-\infty}^{+\infty}\upsilon^2 d\xi = -\varepsilon\left[\int_{-\infty}^{+\infty}\upsilon(\xi)\,d\xi\right]^2. \qquad (8)$$

Substituting here soliton solution (4), we derive the equation for the parameter B:

$$\frac{dB}{d\tau} = \varepsilon\sqrt{\frac{-3\mu}{2\mathrm{Ur}}}\,\frac{B}{\sqrt{1-B^2}}\ln^2\frac{1-\sqrt{1-B^2}}{1+\sqrt{1-B^2}}. \qquad (9)$$

After separation of variables the analytical solution to this equation can be presented in the implicit form in the quadrature. In the KdV limit ($B_0 \to 1$, $\mu \to 0$, $\mathrm{Ur} \to 12$) solution to equation (9) can be found in the explicit form: $B = 1 - (1 - B_0)(1 - \varepsilon\tau)^2$. In terms of soliton amplitude this solution is: $U = U_0(1 - \varepsilon\tau)^2$ [3–6, 9]. As follows from this formula, a soliton completely vanishes in the finite time $\tau_{ext} = 1/\varepsilon$, but actually it transfers after long-term evolution into the envelope soliton [4, 5]. In another limit, when $B_0 \to 0$ ($\mu \to -1$, $\mathrm{Ur} \to 24$), solution of equation (9) again simplifies and in terms of soliton amplitude reduces to $U = U_0\,[1 - 2\exp(-1/\varepsilon\tau)]$. According to this formula, soliton ampli-tude turns to zero at $\tau_{ext} = 1/(\varepsilon\ln 2)$. Hence, the extinction time of the table-top soliton is greater than the extinction time of the KdV soliton by the factor of $1/\ln 2 \approx 1.443$.

In general, equation (9) can be solved numerically and then the solution can be presented in terms of soliton amplitude $U(\tau)$, velocity $V(\tau)$, front width $\Delta(\tau)$, and the total soliton width $D(\tau)$. Time dependence of soliton amplitude is shown in figure 1 for different initial values of the parameter B**.**

The asymptotic dependence for the KdV soliton completely coincides with the numerical solution shown in figure 1 by line 1 for $B_0 = 0.9999$. Another asymptotic solution corresponding to the limiting case of $B_0 = 0$ and $\mu = -1$ is presented in figure 1 by dashed line, whereas dotted line next to line 3 shows the asymptotic solution at finite value of $B_0 = 10^{-4}$. In the latter case there is a good agreement between the asymptotic and numerical solutions if soliton amplitude is not too small.

As one can see from these graphics, soliton amplitude monotonically decrease with time independently of the initial value of the governing parameter B. At a certain time, the amplitude formally vanishes within the framework of the adiabatic theory. The corresponding extinction time has been presented above for two limiting case of the KdV soliton ($B_0 \to 1$) and table-top soliton ($B_0 \to 0$). In general, the extinction time can be found from equation (9) when B turns to unity; then we have:

$$\varepsilon\tau_{ext}(B_0) = \frac{4}{\sqrt{1-B_0^2}}\int_{B_0}^{1}\frac{\sqrt{1-B^2}\,dB}{B\left[\ln\left(1-\sqrt{1-B^2}\right)-\ln\left(1+\sqrt{1-B^2}\right)\right]^2}. \qquad (10)$$

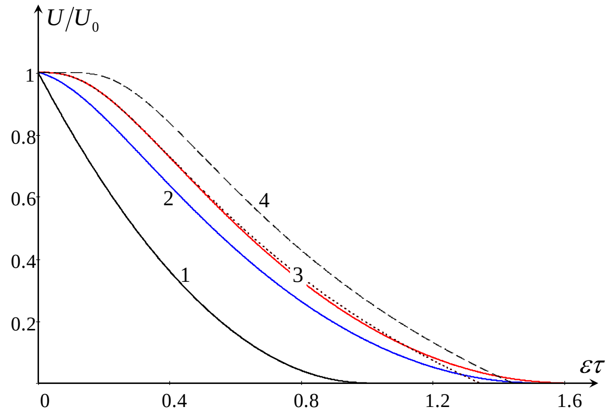

Figure 1. Soliton amplitude against time in normalized variables. Line 1: $B_0 = 0.9999$ (KdV soliton); line 2: $B_0 = 10^{-2}$ ("fat soliton"); solid line 3: $B_0 = 10^{-4}$ (table-top soliton). Dotted line next to line 3 represents the near-asymptotic dependence when $B_0 \to 0$ for $B_0 = 10^{-4}$. Dashed line 4 displays the asymptotic dependence in the limiting case when $B_0 = 0$ and $\mu = -1$.

As follows from this formula, the extinction time goes to infinity when $B_0 \to 0$ (i.e. for table-top solitons), whereas according to the rough estimate it attains a finite value $\varepsilon\tau_{ext} = 1/\ln 2 \approx 1.443$. The minimum value of the extinction time realizes for the KdV soliton.

The soliton velocity is related to its amplitude; the dependence of $V(\tau)$ is shown in figure 2 for the same three initial values of the parameter B as in figure 1.

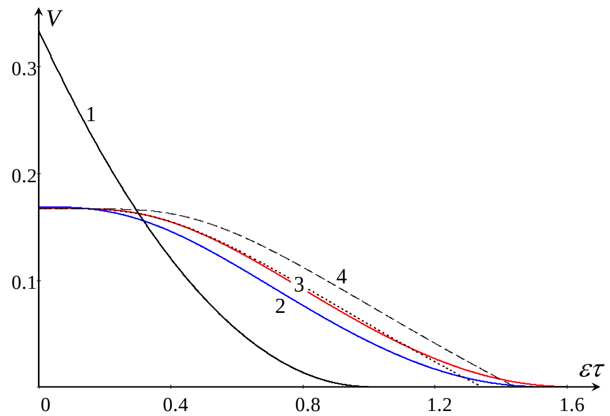

Figure 2. Soliton velocity against time. Line 1: $B_0 = 0.9999$ (KdV soliton); line 2: $B_0 = 10^{-2}$ ("fat soliton"); solid line 3: $B_0 = 10^{-4}$ (table-top soliton). Dotted line next to line 3 corresponds to the asymptotic dependence for the soliton amplitude with $B_0 = 10^{-4}$, and dashed line 4 corresponds to the asymptotic dependence for the soliton amplitude with $B_0 = 0$ and $\mu = -1$.

Figure 2 illustrates that the soliton velocity decreases monotonically with time independently of the initial value of the governing parameter B. The traversed path for the KdV soliton until its disappearance can be easily calculated:

$$\frac{S_{KdV}}{L_0} = \int_0^{\tau_{ext}} V(\tau)\,d\tau = \frac{1}{3}\int_0^{\tau_{ext}}(1-\varepsilon\tau)^2\,d\tau = \frac{1}{9\varepsilon}. \qquad (11)$$

In the case of the table-top soliton with $B_0 = 10^{-4}$ the total traversed path can be evaluated numerically; the result is $S_{tts}/L_0 \approx 0.142/\varepsilon$. For the limiting case of the table-top soliton with $B_0 = 0$ the total traversed path can be calculated analytically; the result is $S_{lim}/L_0 = -4\mathrm{Ei}(-\ln 4)/3\varepsilon \approx 0.159/\varepsilon$, where Ei (x) is the exponential integral function of x.

The width of the soliton front $\Delta(\tau)$ and the total soliton width can be also found in terms of the parameter $B(\tau)$. The total soliton width D can be defined as the distance between the soliton front and rear slopes at the level of half of soliton amplitude, i.e. when $\upsilon(D, \tau) = U/2$ for any instant of time [1]. From equation (4) we derive:

$$\frac{D}{L_0} = \sqrt{-\frac{6\mu}{\mathrm{Ur}}}\frac{4}{\sqrt{1-B^2}}\ln\left[\frac{\sqrt{1+\sqrt{1-B^2}}+\sqrt{1-\sqrt{1-B^2}}+\sqrt{2(1+3B)}}{2\sqrt{B}}\right].$$

In the course of soliton propagation, the width of soliton front monotonically increases with time, whereas the behaviour of the total soliton width may be non-monotonic depending on the initial value of the parameter B. The minimum value of D/L_0 approximately equals to $4.746(-6\mu/\mathrm{Ur})^{1/2}$ and occurs at $B \approx 0.451$; this value is attained at a certain time if the initial soliton amplitude is large enough and $B_0 < 0.451$. Thus, a small-amplitude soliton with $B_0 > 0.451$, whose initial width $D > D_{min}$, decays in time in the course of propagation, and its width monotonically increases with time, whereas large-amplitude soliton with $B_0 < 0.451$, whose initial width $D > D_{min}$, first shrinks in time in the course of propagation, attains the minimal width D_{min}, and only after that expands with time.

Dynamics of Gardner solitons when $\mu > 0$ ($\alpha_1 > 0$)

The physical situations when the cubic nonlinear coefficient α_1 is positive are also possible in real oceanic conditions. Soliton solution of equation (3) with $\varepsilon = 0$ and $\mu > 0$ can be described by the same equation (4) where now $B^2 > 1$. In fact, in this case we have two families of solitons: one for $B \geq 1$ and another for $B \leq -1$. Plots of soliton profiles in terms of $\mu\upsilon$ against $\zeta = (\mathrm{Ur}/6\mu)^{1/2}\xi$ are shown in figure 3 for several values of parameter B.

When $B \to 1_+$, the soliton (4) reduces to the KdV soliton of infinitely small amplitude, which eventually vanishes when B turns to unity. When B increases, the soliton amplitude is also increases and it becomes narrower.

For the negative B solitons are of a negative polarity. Their amplitudes infinitely increase as $B \to -\infty$ and they become narrower. However, when $B \to -1_-$, solitons do not vanish, but reduce to the algebraic soliton shown by line 5 in figure 3. The limiting formula for the algebraic soliton with $B = -1$ is:

$$\upsilon = \frac{U_a}{1+\xi^2/D_a^2} \quad \text{where} \quad U_a = -\frac{2}{\mu}, \quad D_a = \frac{6\mu}{\mathrm{Ur}}.$$

When $\varepsilon \neq 0$ in equation (3), but sufficiently small, $\varepsilon \ll 1$, the asymptotic approach based on the energy balance equation (8) can be developed again. This leads to the equation for the parameter B (cf. equation (9)):

$$\frac{dB}{d\tau} = -8\varepsilon\sqrt{\frac{6\mu}{\mathrm{Ur}}}\frac{B}{\sqrt{B^2-1}}\arctan^2\sqrt{\frac{B-1}{B+1}}. \tag{12}$$

The solution to this equation can be presented in the quadrature. In the KdV limit ($B_0 \to 1_+$), equation (12) simplifies and reduces to the very same equation which was presented above. In another limit $B_0 \to \pm\infty$ equation (12) again simplifies and reduces to

$$\frac{dB}{d\tau} = -\frac{\varepsilon\pi^2}{4}B_0\left(1-\frac{4}{\pi B}\right). \tag{13}$$

Integrating this equation we obtain the implicit dependence $B(\tau)$:

$$\varepsilon\tau = \frac{\pm 4}{\pi^2 B_0}\left(\frac{4}{\pi}\ln\frac{B_0}{B} + B_0 - B\right). \tag{14}$$

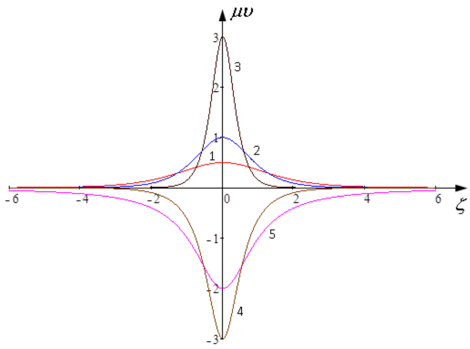

Figure 3. Normalised Gardner soliton (4) with $\mu > 0$, for several values of parameter B. Line 1: $B = 1.5$; line 2: $B = 2$; line 3: $B = 4$; line 4: $B = -2$; line 5: $B = -1$ (the algebraic soliton).

Here sign plus (minus) corresponds to the case of $B \to +\infty$ ($B \to -\infty$). This formula can be further simplified for $B \approx B_0$; then we have $B = B_0(1 - \pi^2\varepsilon\tau/4)$. In terms of soliton amplitude this gives $U = U_0(1 - \pi^2\varepsilon\tau/4)$.

These formulae make sense only until $|B| \geq 1$. When B is positive and decreasing from some value $B_0 > 1$ to $B = 1$, the soliton gradually vanishes reducing first into the KdV soliton which completely vanishes then in finite time. Figure 4 illustrates this process in terms of soliton amplitude versus normalized time. Solid lines 1, 2 and 3 in this figure represent numerical solutions of equation (12) for different values of the parameter B_0.

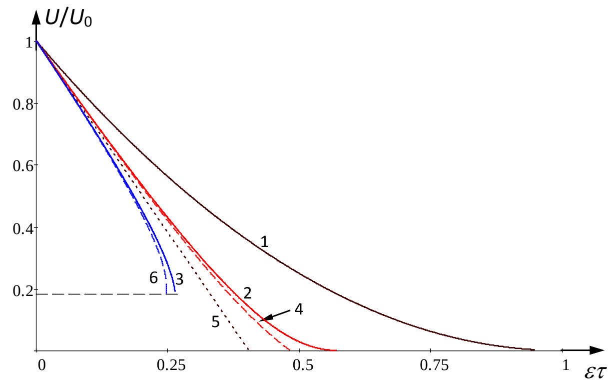

Figure 4. Bell-shaped soliton amplitude against time in normalized variables. Line 1: $B_0 = 1.01$ (quasi-KdV soliton); line 2: $B_0 = 10$; line 3: $B_0 = -10$. Dashed line 4 represents the asymptotic dependence (14) for $B_0 = 10$. Dotted line 5 displays the limiting case $U = U_0(1 - \pi^2\varepsilon\tau/4)$ when $B \approx B_0$; and dashed line 6 represents the asymptotic dependence (14) for $B_0 = -10$.

The higher the parameter B_0, the faster the soliton decays (cf. lines 1 for $B_0 = 1.01$ and 2 for $B_0 = 10$). The shortest life-time of the bell-shaped solitons with $B_0 > 1$ can be roughly estimated from the limiting formula valid for $B \approx B_0$. According to that formula, soliton amplitude turns to zero at $\tau = \tau_{ext} \equiv 4/(\varepsilon\pi^2)$, which is less than the extinction time of the KdV soliton in factor $4/\pi^2$. The analytical dependence for the KdV soliton with $B_0 = 1$ is indistinguishable from numerically obtained line 1 in figure 4.

The situation is different when $B_0 < -1$. In this case the adiabatic theory predicts that a soliton decays until its parameter B increases, but remains less than –1. However, when B becomes equal to –1, the soliton does not vanish, but transforms into the algebraic soliton. As has been shown [10], the algebraic soliton is structurally unstable, i.e. under small perturbations it reduces to the breather – non-stationary solitary wave with the oscillating internal structure [9]. Thus, under the action of Earth's rotation the Gardner soliton of negative polarity decays and reduces to the algebraic soliton of amplitude $U_{\lim} = 2U_0(1 - B_0)$. Apparently the breather further decays, but its evolution should be studied separately.

Dotted line 5 in figure 4 separates the decay lines of solitons with positive polarity (the corresponding lines lay to the right of line 5) and solitons with negative polarity (their decay lines lay to the left of line 5). The decay lines of solitons with negative polarity terminate at the finite values $U_{\lim}$ when the parameter B becomes equal to –1 (see horizontal dashed line in figure 4 for $B_0 = -10$).

When equation (12) is solved for the parameter B, all other soliton parameters (U, V, Δ) can be readily obtained as functions of τ by means of equations (5) and (6). As has been mentioned above, the soliton amplitude formally vanishes at a certain time, if $B_0 > 1$, or reduces to $U_{\lim}$, if $B_0 < -1$. The corresponding extinction time can be found from equation (12) when B turns to ±1; then we have:

$$\varepsilon\tau_{ext}\left(B_0\right) = -\frac{1}{8}\sqrt{\frac{\mathrm{Ur}}{6\mu}}\int_{B_0}^{\pm 1}\frac{\sqrt{B^2-1}\,dB}{B\arctan^2\sqrt{\dfrac{B-1}{B+1}}}. \tag{15}$$

Figure 5 shows the dependences of extinction time on B_0. In the same figure we present a dependence of the extinction time for the fat solitons as per equation (10) (see line 1). As follows from this figure, line 2 represents just a continuous sequential of line 1, and the extinction time for the KdV soliton $\varepsilon\tau_{ext} = 1$ exactly corresponds to the point of matching of lines 1 and 2 (see the black dot between lines 1 and 2 in figure 5).

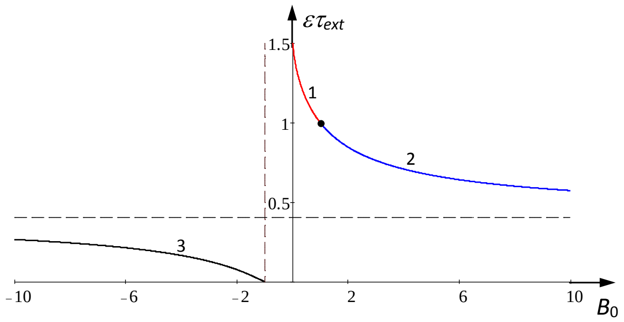

Figure 5. Extinction time for all types of Gardner solitons against the initial value of parameter B. Line 1: fat solitons when $\mu = -1$; line 2: bell-shaped solitons with $B_0 > 1$ when $\mu = 1$; line 3: bell-shaped solitons with $B_0 < -1$ when $\mu = 1$. Black dot corresponds to the KdV soliton, and dashed horizontal line shows the asymptotic value of the extinction time $4/\pi^2$ for $B_0 \to \pm\infty$.

Thus, the adiabatic theory predicts that the extinction time for fat and table-top solitons (when $\mu < 0$) is always greater than the extinction time of the KdV soliton, whereas the extinction time of the bell-shaped solitons (when $\mu > 0$) is always less than the extinction time of the KdV soliton. Further, the characteristic time of bell-shaped solitons transformation into the algebraic soliton (when $B_0 < -1$) is always less than the extinction time of bell-shaped solitons with $B_0 > 1$ (cf. lines 2 and 3 in figure 5). When $B_0 \to \pm\infty$, the extinction time of bell-shaped solitons asymptotically approach a limiting value $4/\pi^2$, but from different sides (see the dashed horizontal line $\varepsilon\tau_{ext} = 4/\pi^2$ in figure 5 and lines 2 and 3 approaching to it).

Conclusions

Thus, the adiabatic decay of different types of internal wave solitons were calculated within the framework of the Gardner–Ostrovsky equation. It was shown that at the early stage of evolution solitons gradually decay under the influence of weak Earth' rotation which provides the additional dispersive term in the Gardner equation. The characteristic decay time was derived for the various types of solitons (table-top, fat, KdV and bell-shaped solitons of different polarity), which can exist within the Gardner equation (see [1, 9] and references therein). One can expect that in the long-term evolution Gardner solitons eventually transform into the envelope solitons described by the nonlinear Schrödinger equation. The similar transformation of KdV solitons within the Ostrovsky equation is well-known [4, 5]. The long-term evolution of Gardner solitons has been studied in Ref. [13] for the particular cases; however a further study is still required to clarify the asymptotic state of soliton evolution as well as some other related issues.

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